METHOD OF COMPUTATION OF THE ABSOLUTE ANGULAR VELOCITY AND ABSOLUTE ANGULAR ACCELERATION IN CASE OF COMPOUND MOTION OF THE SOLID RIGID BODY

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Abstract

This paper presents in a very detailed manner the proof of the calculus formula which is used for the computation of the absolute angular velocity and the absolute angular acceleration in the case of the compound motion of the solid rigid body. The relative angular velocity relatively to the movable reference frame and the movement of the mobile frame relatively to the fixed reference frame are considered to be known. Further on the absolute angular velocity and the absolute angular acceleration are to be calculated. The method which is presented in the paper is based on matrix calculus.

Keywords: absolute angular speed, absolute angular acceleration, compound motion, movable frame

1. INTRODUCTION

We will consider a fixed reference frame $T_0(O_0x_0y_0z_0)$, a movable reference frame $T_1(O_1x_1y_1z_1)$ and a body fixed reference frame $T_2O_2x_2y_2z_2)$ which are shown in the figure no.1. The solid rigid body position relatively to the frame $T_1(O_1x_1y_1z_1)$ is specified by the relative coordinates $x_{O_{2,1}}$, $y_{O_{2,1}}$ $z_{O_{2,1}}$ of the body fixed reference frame $T_2O_2x_2y_2z_2)$ origin O_2 and by Euler's angels Ψ_{21}, θ_{21}, and φ_{21} .The position vector of the body fixed reference frame origin O_2 relatively to the movable frame $T_1(O_1x_1y_1z_1)$has the following matrix expression:

$$\{O_1O_2\} = \{r_{O_{21}}\} = \left[x_{O_{2,1}} \mid y_{O_{2,1}} \mid z_{O_{2,1}}\right]^T \qquad (1)$$

The Euler's angels Ψ_{21}, θ_{21}, and φ_{21} may be written in a single column vector as following:

$$\{q_{r_{21}}\} = [\Psi_{21} \mid \theta_{21} \mid \varphi_{21}]^T \qquad (2)$$

The position of the movable frame $T_1(O_1x_1y_1z_1)$ relatively to the fixed reference frame $T_0(O_0x_0y_0z_0)$ is established by the coordinates of its origin $O_1, x_{O_{10}}$, $y_{O_{10}}, z_{O_{10}}$ and by Euler's angels Ψ_{10}, θ_{10}, and φ_{10}.

$$\{O_0O_1\} = \{r_{O_{10}}\} = \left[x_{O_{1,0}} \mid y_{O_{10}} \mid z_{O_{10}}\right]^T \qquad (3)$$

The Euler's angels Ψ_{10}, θ_{10}, and φ_{10} may be written into a single column matrix as following:

$$\{q_{r_{10}}\} = [\Psi_{10} \mid \theta_{10} \mid \varphi_{10}]^T \qquad (4)$$

2. COMPUTATION OF ABSOLUTE ANGULAR VELOCITY

In order to determine the absolute angular velocity of the solid rigid body "S" we will express the absolute velocity of the particle "A" as a sum between its relative velocity and its velocity of conveying as following:

$$\{v_A^a\} = \{v_A^{rel.}\} + \{v_A^{con.}\} \qquad (5)$$

The vector relation (5) written under matrix form is expressed by its projections on the fixed reference frame $T_0(O_0x_0y_0z_0)$. The relative velocity of the material point "A" regarded as a point which belongs to the solid rigid "S" will be written by using the relation which gives us the distribution of velocities:

$$\{v_A^{rel.}\} = [R_{10}] \cdot [R_{21}] \cdot (\{v_{O_{2,1}}\} + [\omega_{21}] \cdot \{O_2A\}) \quad (6)$$

The velocity of conveying will be written in its projections on the fixed reference frame using

Euler's formula for distribution of velocities as following:

$$\{v_A^{conv.}\} = [R_{10}] \cdot (\{v_{O_{10}}\} + [\omega_{10}] \cdot \{O_1A\}) \quad (7)$$

In the relations (6) and (7) the terms have the following expressions:

$$[R_{10}] = [\Psi_{10}] \cdot [\Theta_{10}] \cdot [\Phi_{10}] \quad (8)$$

$$[R_{21}] = [\Psi_{21}] \cdot [\Theta_{21}] \cdot [\Phi_{21}] \quad (9)$$

$$\{O_1A\} = \{O_1O_2\} + [R_{21}] \cdot \{O_2A\} \quad (10)$$

$$[\Psi_{10}] = \begin{bmatrix} \cos(\Psi_{10}) & -\sin(\Psi_{10}) & 0 \\ \sin(\Psi_{10}) & \cos(\Psi_{10}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

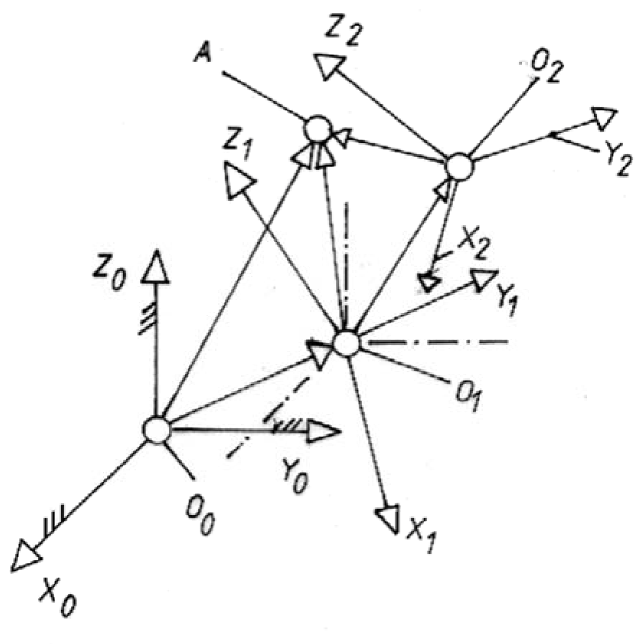

Fig.1 – This figure presents the motion of a solid rigid body relatively to the movable frame $T_1(O_1x_1y_1z_1)$ and the motion of the movable frame $T_1(O_1x_1y_1z_1)$ relatively to the fixed frame $T_0(O_0x_0y_0z_0)$

$$[\Theta_{10}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{10}) & -\sin(\theta_{10}) \\ 0 & \sin(\theta_{10}) & \cos(\theta_{10}) \end{bmatrix} \quad (12)$$

$$[\Phi_{10}]=\begin{bmatrix}\cos(\varphi_{10}) & -\sin(\varphi_{10}) & 0\\ \sin(\varphi_{10}) & \cos(\varphi_{10}) & 0\\ 0 & 0 & 1\end{bmatrix} \quad (13)$$

$$[\Psi_{21}]=\begin{bmatrix}\cos(\Psi_{21}) & -\sin(\Psi_{21}) & 0\\ \sin(\Psi_{21}) & \cos(\Psi_{21}) & 0\\ 0 & 0 & 1\end{bmatrix} \quad (14)$$

$$[\Theta_{21}]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_{21}) & -\sin(\theta_{21})\\ 0 & \sin(\theta_{21}) & \cos(\theta_{21})\end{bmatrix} \quad (15)$$

$$[\Phi_{21}]=\begin{bmatrix}\cos(\varphi_{21}) & -\sin(\varphi_{21}) & 0\\ \sin(\varphi_{21}) & \cos(\varphi_{21}) & 0\\ 0 & 0 & 1\end{bmatrix} \quad (16)$$

The absolute velocity of the particle "A" may be written in projections on the axis of the fixed frame thus:

$$\{v_A^a\}=[R_{10}]\cdot[R_{21}]\cdot\left(\{v_{O_{2,0}}\}+[\omega_{20}]\cdot\{O_2A\}\right) \quad (17)$$

We will replace the relations (6) and (7) into the relation (17) and we will obtain:

$$[A]=[B]+[C] \quad (18)$$

$$[A]=[R_{10}]\cdot[R_{21}]\cdot[\omega_{20}]\cdot\{O_2A\} \quad (19)$$

$$[B]=[R_{10}]\cdot[R_{21}]\cdot[\omega_{21}]\cdot\{O_2A\} \quad (20)$$

$$[C]=[R_{10}]\cdot[\omega_{21}]\cdot[R_{21}]\cdot\{O_2A\} \quad (21)$$

The relation (18) may be written in an equivalent form as following:

$$\{\omega_{20}\}=[R_{21}]^T\cdot\{\omega_{10}\}+\{\omega_{21}\} \quad (22)$$

We will multiply the relation (22) to the left with the matrix $[R_{21}]$ and we will obtain:

$$[R_{21}]\cdot\{\omega_{20}\}=\{\omega_{10}\}+[R_{21}]\cdot\{\omega_{21}\} \quad (23)$$

3. COMPUTATION OF ABSOLUTE ANGULAR ACCELERATION

In order to determine the absolute angular acceleration we will derive the relation (23) with respect to time and we will obtain:

$$[A_1]=[B_1]+[C_1] \quad (24)$$

In the relation (24) the terms $[A_1]$, $[B_1]$ and $[C_1]$ have the following expressions:

$$[A_1]=[\dot{R}_{21}]\cdot\{\omega_{20}\}+[R_{21}]\cdot\{\dot{\omega}_{20}\} \quad (25)$$

$$[B_1]=\{\dot{\omega}_{10}\} \quad (26)$$

$$[C_1]=[\dot{R}_{21}]\cdot\{\omega_{21}\}+[R_{21}]\cdot\{\dot{\omega}_{21}\} \quad (27)$$

In the relations (25), (26) and (27) the point "•" denotes the derivation with respect to time:

$$[\dot{R}_{21}]=d([R_{21}])/dt \quad (28)$$

$$\{\dot{\omega}_{20}\}=d(\{\omega_{20}\})/dt \quad (29)$$

$$\{\dot{\omega}_{21}\}=d(\{\omega_{21}\})/dt \quad (30)$$

We will multiply the relation (24) to the left with the matrix $[R_{21}]^T$ and we will obtain:

$$[A_2]=[B_2]+[C_2] \quad (31)$$

In the relation (31) the terms $[A_2]$, $[B_2]$ and $[C_2]$ have the following expressions:

$$[A_2]=\{\dot{\omega}_{20}\}+[\omega_{21}]\cdot\{\omega_{20}\} \quad (32)$$

$$[B_2]=[R_{21}]^T\cdot\{\dot{\omega}_{10}\} \quad (33)$$

$$[C_2]=\{\dot{\omega}_{21}\} \quad (34)$$

$$[\omega_{21}]=[R_{21}]^T\cdot[\dot{R}_{21}] \quad (35)$$

$$[\omega_{21}]=\begin{bmatrix}0 & -\omega_{21\_z_2} & \omega_{21\_y_2}\\ \omega_{21\_z_2} & 0 & -\omega_{21\_x_2}\\ -\omega_{21\_y_2} & \omega_{21\_x_2} & 0\end{bmatrix} \quad (36)$$

$$\omega_{21\_x_2}=\dot{\Psi}_{21}\cdot\sin(\theta_{21})\cdot\sin(\varphi_{21})+\dot{\theta}_{21}\cdot\cos(\varphi_{21}) \quad (37)$$

$$\omega_{21\_y_2}=\dot{\Psi}_{21}\cdot\sin(\theta_{21})\cdot\cos(\varphi_{21})-\dot{\theta}_{21}\cdot\sin(\varphi_{21}) \quad (38)$$

$$\omega_{21\_z_2}=\dot{\Psi}_{21}\cdot\cos(\theta_{21})+\dot{\varphi}_{21} \quad (39)$$

It is well-known that:

$$\{\omega_{20}\} = [R_{21}]^T \cdot \{\omega_{10}\} + \{\omega_{21}\} \quad (40)$$

We will replace the relation (40) into relation (31) and we will obtain:

$$[A_3] = [B_3] + [C_3] \quad (41)$$

In the relation (41) the terms $[A_3]$, $[B_3]$ and $[C_3]$ have the following expressions:

$$[A_3] = \{\dot{\omega}_{20}\} \quad (42)$$

$$[B_3] = [R_{21}]^T \cdot \{\dot{\omega}_{10}\} \quad (43)$$

$$[C_3] = \{\dot{\omega}_{21}\} - [\omega_{21}] \cdot [R_{21}]^T \cdot \{\omega_{10}\} \quad (44)$$

The relation (41) will be multiplied to the left with the matrix $[R_{21}]$ and we will obtain:

$$[A_4] = [B_4] + [C_4] \quad (45)$$

In the relation (45) the terms $[A_4]$, $[B_4]$ and $[C_4]$ have the following expressions:

$$[A_4] = [R_{21}] \cdot \{\dot{\omega}_{20}\} \quad (46)$$

$$[B_4] = \{\dot{\omega}_{10}\} \quad (47)$$

$$[C_4] = [R_{21}] \cdot \{\dot{\omega}_{21}\} + [\omega_{10}] \cdot [R_{21}] \cdot \{\omega_{21}\} \quad (48)$$

$$[\omega_{10}] = \begin{bmatrix} 0 & -\omega_{10\_z_1} & \omega_{10\_y_1} \\ \omega_{10\_z_1} & 0 & -\omega_{10\_x_1} \\ -\omega_{10\_y_1} & \omega_{10\_x_1} & 0 \end{bmatrix} \quad (49)$$

$$\omega_{10\_x_1} = \dot{\Psi}_{10} \cdot \sin(\theta_{10}) \cdot \sin(\varphi_{10}) + \dot{\theta}_{10} \cdot \cos(\varphi_{10}) \quad (50)$$

$$\omega_{10\_y_1} = \dot{\Psi}_{10} \cdot \sin(\theta_{10}) \cdot \cos(\varphi_{10}) - \dot{\theta}_{10} \cdot \sin(\varphi_{10}) \quad (51)$$

$$\omega_{10\_z_1} = \dot{\Psi}_{10} \cdot \cos(\theta_{10}) + \dot{\varphi}_{10} \quad (52)$$

$$\{\dot{\omega}_{10}\} = \left[\dot{\omega}_{10\_x_1} \;\vdots\; \dot{\omega}_{10\_y_1} \;\vdots\; \dot{\omega}_{10\_z_1}\right]^T \quad (53)$$

$$\{\dot{\omega}_{20}\} = \left[\dot{\omega}_{20\_x_2} \;\vdots\; \dot{\omega}_{20\_y_2} \;\vdots\; \dot{\omega}_{20\_z_2}\right]^T \quad (54)$$

$$\begin{aligned}\dot{\omega}_{10\_x_1} &= \ddot{\Psi}_{10} \sin(\theta_{10})\sin(\varphi_{10}) + \ddot{\theta}_{10} \cos(\varphi_{10}) + \\ &+ \{\dot{q}_{r\_10}\}^T [A] \{\dot{q}_{r\_10}\}\end{aligned} \quad (55)$$

$$\begin{aligned}\dot{\omega}_{10\_y_1} &= \ddot{\Psi}_{10} \sin(\theta_{10})\cos(\varphi_{10}) - \ddot{\theta}_{10} \sin(\varphi_{10}) + \\ &+ \{\dot{q}_{r\_10}\}^T [B] \{\dot{q}_{r\_10}\}\end{aligned} \quad (56)$$

$$\dot{\omega}_{10\_z_1} = \ddot{\Psi}_{10} \cos(\theta_{10}) + \ddot{\varphi}_{10} + \{\dot{q}_{r\_10}\}^T [C] \{\dot{q}_{r\_10}\} \quad (57)$$

$$\{\dot{q}_r\} = \left[\dot{\Psi} \;\vdots\; \dot{\theta} \;\vdots\; \dot{\varphi}\right]^T \quad (58)$$

$$[A] = \begin{bmatrix} 0 & (1/2)\cos(\theta)\sin(\varphi) & A_{13} \\ A_{21} & 0 & -(1/2)\sin(\varphi) \\ A_{31} & A_{32} & 0 \end{bmatrix} \quad (59)$$

$$[B] = \begin{bmatrix} 0 & B_{12} & B_{13} \\ B_{21} & 0 & -(1/2)\cos(\varphi_{10}) \\ B_{31} & B_{32} & 0 \end{bmatrix} \quad (60)$$

$$[C] = \begin{bmatrix} 0 & -(1/2)\sin(\theta_{10}) & 0 \\ -(1/2)\sin(\theta_{10}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (61)$$

$$A_{21} = (1/2)\cos(\theta)\sin(\varphi) \quad (62)$$

$$A_{13} = A_{31} = (1/2)\sin(\theta)\cos(\varphi) \quad (63)$$

$$A_{32} = -(1/2)\sin(\varphi) \quad (64)$$

$$B_{21} = (1/2)\cos(\theta)\cos(\varphi) \quad (65)$$

$$B_{31} = B_{13} = -(1/2)\sin(\theta)\sin(\varphi) \quad (66)$$

$$B_{32} = -(1/2)\sin(\varphi) \quad (67)$$

The relation (45) gives us the formula of calculus of the absolute angular acceleration. This formula will be applied whenever we are dealing with the compound motion of a solid rigid body.

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